Coxeter Groups and Complexes

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1 Introduction

Let W be a group which is generated by a set S of order 2 elements. We will often denote this pair, or "system" as (W, S). If $s, t \in S$, we denote the order of st as $m(s, t)$.

Definition 1.1 (W, S) *is a* ***Coxeter system*** *if* W *admits the presentation*

$$\Big\langle S\ ;\ (st)^{m(s,t)} = 1 \Big\rangle. \quad (1)$$

where $s, t \in S$*,* $m(s, t) < \infty$ *and there is one relation for each pair* s, t*. We accordingly refer to* W *as a* ***Coxeter group****, and the condition described in the presentation is referred to as the* ***Coxeter condition****.*

Such systems will be the focus of this paper. Although our initial results are developed by purely algebraic considerations of (W, S), we will eventually explore the rich geometry associated to Coxeter systems.

1.1 Element Decompositions

For the remainder of this paper, we assume W is a Coxeter group. We begin by considering elements of W as decompositions of elements in S, i.e., $w = s_1 s_2 \dots s_d$ where $s_i \in S$. We will often refer to the generators in such a decomposition as *letters.*

Definition 1.2 *If* $w \in W$*, and* $w = s_1 s_2 \dots s_n$ *where* $s_i \in S$*, is a reduced composition of* w*, that is, there is no shorter decomposition representing* w*, then we set the* ***length*** *of* w*,* $l(w) = n$*.*

An equivalent form of the Coxeter condition is the following,

Deletion Condition *Let $w \in W$. If $w = s_1 s_2 \dots s_k$ with $k > l(w)$, there are indices $i < j$ such that $w = s_i \dots \hat{s}_i \dots \hat{s}_j \dots s_k$, where $\hat{s}_i$ and $\hat{s}_j$ are deleted letters from our original decomposition.*

We will often use this hat notation to denote deleted letters.

Definition 1.3 *For $J \subseteq S$, we denote $W_J = \langle J \rangle$. We refer to W_J as a standard subgroup. For $w \in W$, we refer to wW_J as a standard coset.*

For $J \subseteq S$, (W_J, J) is a Coxeter system in its own right, which follows from Lemma 1.4 below. Standard subgroups and cosets will be of immense importance in developing a geometric interpretation of Coxeter systems in the second section.

Let W_J be a standard subgroup, where $J \subseteq S$. For $w \in W_J$, if $w = s_1 s_2 \dots s_d$, where $s_i \in J$, is a reduced composition of w, that is, there is no shorter decomposition representing w inside W_J, then we set the length of w with respect to W_J, $l_J(w) = d$.

Lemma 1.4 *If $w \in W_J$ for some $J \subseteq S$, then*

$$l_J(w) = l(w)$$

Proof. Let $w \in W_J$ and $w = s_1 s_2 \dots s_d$ be a reduced J-decomposition of w. If there is a shorter S-decomposition of w then by the Deletion Condition, we can obtain one by deleting two letters from our J-decomposition, but then this would contradict our J-decomposition being reduced. ∎

We have the following relation between the length of a word and the letters used in a given reduced decomposition.

Proposition 1.5 *For $w \in W$, there is a subset $S(w) \subseteq S$ such that any reduced decomposition of w uses precisely the letters in $S(w)$, and $S(w)$ is the smallest subset of S whose associated standard subgroup contains w.*

Proof. See [1], Proposition 2.16. ∎

1.2 Simplicial Complexes

Let $\mathcal{V}$ be nonempty set. We will refer to $\mathcal{V}$ as a *vertex set* and its elements as *vertices*. Informally, a *simplicial complex* with vertex set $\mathcal{V}$ is a nonempty collection Δ of finite subsets of $\mathcal{V}$, called *simplices*, such that every singleton (which are associated to vertices) is a simplex and every subset of a simplex is also a simplex. We refer to a subset of a simplex A as a *face* of A.

A *subcomplex* is a collection of subsets in Δ which is a simplicial complex in its own right, with the vertex set being some subset of $\mathcal{V}$. We may give Δ a partial order by the inclusion relation, i.e., for $A, B \in \Delta$, $A \leq B$ if A is a subset (or face) of B. We often refer to such simplicial complexes with inclusion-ordering as *classicial simplicial complexes.* As a poset, we have that Δ satisfies the following properties:

(a) Any two $A, B \in \Delta$ have a greatest lower bound, denoted $A \cap B$.

(b) For $A \in \Delta$, the poset $\Delta_{\leq A}$ of the faces of A is isomorphic to $2^{\{1,2,\dots,r\}}$, the power set of $\{1, 2, \dots, r\}$, for some positive integer r and with inclusion-ordering.

With these two conditions, we may give a formal definition.

Definition 1.6 *Any poset Δ satisfying the properties (a) and (b) is referred to as a* ***simplicial complex****. A* ***subcomplex*** *of Δ is a collection of elements in Δ which form a simplicial complex under the ordering of Δ. If $A \in \Delta$, and $\Delta_{\leq A} \cong 2^{\{1,2,\dots,r\}}$ for some positive integer r, we denote rank$(A) = r$ and dim$(A) = r - 1$.*

For those posets which are not inclusion-ordered on some power set, these are often referred to as *abstract simplicial complexes.* We will convince ourselves that this formal definition can be related to our informal formulation of a simplicial complex.

Let Δ be a poset satisfying conditions (a) and (b) and denote $\mathcal{V} = \{A \in \Delta \mid \text{rank}(A) = 1\}$.

Define

$$\varphi : \Delta \to 2^{\mathcal{V}}$$

where $\varphi(B) = \{A \in \mathcal{V} \mid A \leq B\}$. We will establish several key properties of this map.

Proposition 1.7 *Let φ be as defined above. Then*

1. *$\varphi(B)$ is finite $\forall B \in \Delta$.*
2. *$B \leq C \Leftrightarrow \varphi(B) \subseteq \varphi(C)$.*
3. *$S = \{\varphi(B) \mid B \in \Delta\}$ is a classical simplicial complex.*

Proof. To see 1, note for that any $B \in \Delta$ by condition (b), $|\Delta_{\leq B}|$ is finite, hence the number of vertices in $\Delta_{\leq B}$ is finite.

For 2, we begin with the forward direction. Let $A \in \mathcal{V}$ such that $A \leq B$. If $B \leq C$, by transitivity $A \leq C$, thus $\varphi(B) \subseteq \varphi(C)$. For the other direction, we first let $\varphi(B) = \varphi(C)$ and consider $B \cap C$, the greatest lower bound of B and C. For $A \in \mathcal{V}$, if $A \in \varphi(B) = \varphi(C)$, then $A \leq B$ and $A \leq C$, hence $A \leq B \cap C$. Hence, $\varphi(B) \subseteq \varphi(B \cap C)$, and by the forward direction since $B \cap C \leq B$, then

$\varphi(B \cap C) \subseteq \varphi(B)$. Thus, by antisymmetry $\varphi(B \cap C) = \varphi(B)$. By a very similar argument, $\varphi(B \cap C) = \varphi(C)$. Hence, $\operatorname{rank}(B \cap C) = \operatorname{rank}(B)$. By inspection of the poset $2^{\{1,2,\dots,r\}}$ (inclusion-ordered) isomorphic to $\Delta_{\leq B}$, the integer singletons correspond $A \in \mathcal{V}$ such that $A \leq B$, and there is a unique largest element which contains all of them– that is, B is the only face of B with rank r, hence $B \cap C = B$. By a similar argument $B \cap C = C$, thus $B = C$ and φ is injective.

Now assume $\varphi(B) \subseteq \varphi(C)$, and let $B \cap C$ be the greatest lower bound of B and C. We have already established that $\varphi(B \cap C) \subseteq \varphi(B)$. If $A \in \varphi(B)$, then $A \in \varphi(C)$, so $A \leq B$ and $A \leq C$, hence $A \leq B \cap C$. So, $\varphi(B) = \varphi(B \cap C)$, and by the injectivity of φ, $B = B \cap C$. Thus, $B = B \cap C \leq C$, giving us the desired inclusion.

Lastly, we show 3, that is, $S = \{\varphi(B) \mid B \in \Delta\}$ is a classical simplicial complex. Every singleton, corresponding to some $A \in \mathcal{V}$, is an element of S, since $\varphi(A) = \{A\}$. To see this, note that by definition of rank, $\Delta_{\leq A}$ is isomorphic to the power set of a single integer (inclusion-ordered), so the only rank 1 element less than or equal to A is itself. Now fix $C \in \Delta$. We will show that every subset of $\varphi(C)$ is an element of S. We will look at the restriction map

$$\varphi|_{\Delta_{\leq C}} : \Delta_{\leq C} \to 2^{\mathcal{V}_C}$$

where $\mathcal{V}_C = \{A \in \mathcal{V} \mid A \leq C\}$. By condition (b), $\Delta_{\leq C}$ is isomorphic to the inclusion-ordered power set $2^{\{1,2,\dots,j\}}$ for some positive integer j. Since the integer singletons correspond to rank 1 elements less than or equal to C, $\left|2^{\mathcal{V}_C}\right| = \left|2^{\{1,2,\dots,j\}}\right|$. Since φ is injective, so is the restriction, so we have then that $\varphi|_{\Delta_{\leq C}}$ is bijective. We have already proven that φ (and hence its restriction) preserve ordering when the codomain is endowed with the inclusion-ordering, thus we have a poset isomorphism $\Delta_{\leq C} \cong 2^{\mathcal{V}_C}$ under the restriction mapping, so for $J \in 2^{\mathcal{V}_C}$, there exists $D \in \Delta_{\leq C}$ such that $\varphi|_{\Delta_{\leq C}}(D) = J$, so $\varphi(D) = J$, which proves S is a classical simplicial complex. ∎

1.3 Chamber Complexes

Definition 1.8 *Let Δ be a finite-dimensional simplicial complex. Assume all maximal simplices have the same dimension. A **gallery** is a sequence of maximal simplices such that any two consecutive simplices are **adjacent**, meaning they are distinct and have a common codimension-1 face.*

A *pregallery* allows for consecutive simplices to be equal – this notion will be particularly important later on.

Definition 1.9 *Let Δ be a finite-dimensional simplicial complex. We refer to Δ as a **chamber complex** if all maximal simplices have the same dimension and any two maximal simplices may be connected by a gallery.*

The maximal simplices are referred to as *chambers* and the codimension-1 face of a chamber is called a *panel*. We say a chamber complex is *thin* if every panel is the face of exactly two chambers. We will often denote the chambers of Δ as $\mathcal{C}$ or $\mathcal{C}(\Delta)$. We may give $\mathcal{C}$ a well-defined *distance function* $d(-,-)$ which is set as the length of a minimal gallery between two chambers.

Definition 1.10 *Let Δ be a chamber complex of rank n and let I be a set of n elements (referred to as colors). A **type function** τ on Δ is a map from the vertices (rank-1 elements) of Δ to I such that the restriction to the set of vertices of each chamber is bijective. The **type** of a vertex $v \in \Delta$ is $\tau(v)$. A chamber complex is **colorable** if it admits a type function.*

If τ and τ' are type functions on Δ, then they may only differ in their associated coloring sets I and I', between which there is a bijection. In other words, if Δ is colorable, the color-partition of vertices is unique.

A *simplicial map* ϕ from one simplicial complex Δ to another Δ' is a function ϕ from the vertices of Δ to the vertices of Δ' which takes simplices to simplices. We say ϕ is *nondegenerate* if for any $A \in \Delta$, $\dim(A) = \dim\,(\phi(A))$.

Definition 1.11 *If Δ and Δ' are chamber complexes of the same dimension, we call a simplicial map $\phi : \Delta \to \Delta$ a **chamber map** if it takes chambers to chambers or equivalently, if ϕ is nondegenerate.*

Chamber maps take adjacent chambers to adjacent or equal chambers, so they take galleries to pregalleries.

A *chamber subcomplex* Σ of a chamber complex Δ is a simplicial subcomplex which is a chamber complex in its own right, and has the same dimension as Δ. If $\phi : \Delta \to \Sigma$ is a chamber map which acts as the identity on Σ, then ϕ is referred to as a *retraction* of Δ onto Σ, and Σ is called a *retract* of Δ .

We may now extend our definition of type functions. Let Δ be a chamber complex of rank n and I an n-element set. We may define a type function (and chamber map) $\tau : \Delta \to 2^I$, where the codomain is the power set of I ordered by inclusion and for $A \in \Delta$, $\tau(A)$, called the *type* of A, consists of the types of vertices which are faces of A. The *cotype* of A is $I \setminus \tau(A)$.

Proposition 1.12 *Let Δ and Δ' be colorable chamber complexes with respective type functions τ and τ' and respective color sets I and I'. If $\phi : \Delta \to \Delta'$ is a chamber map, then there is a bijection $\phi_* : I \to I'$ such that for $A \in \Delta$,*

$$\tau'(\phi(A)) = \phi_*(\tau(A)).$$

Proof. See [1], Proposition A.14. ■

2 Coxeter Complexes

We may use the theory developed in the first section to gain a geometric flavor of Coxeter standard cosets. In particular, we will associate elements of a Coxeter group W with chambers of a simplicial complex consisting of standard cosets. This will allow us to prove certain properties which are otherwise very difficult to show with purely algebraic arguments.

2.1 $\Sigma(W, S)$

Definition 2.1 *Let (W, S) be a Coxeter system. We denote by $\Sigma := \Sigma(W, S)$ the poset of standard of cosets in W, ordered by reverse inclusion. That is, $B \leq A$ in Σ if and only if $B \supseteq A$ when regarding these cosets as subsets of W, and we call B a* ***face*** *of A. We refer to $\Sigma(W, S)$ as the* ***Coxeter complex*** *associated to (W, S).*

Definition 2.2 *Let $\Sigma(W, S)$ be a Coxeter complex. We refer to the elements of Σ as* ***simplices****. The maximal simplices $\{w\}$ where $w \in W$, are called* ***chambers****. Chambers are clearly associated with elements of W. The standard cosets $w\langle s\rangle$, where $w \in W$ and $s \in S$ are called* ***panels****. The* ***fundamental chamber*** *is set as $C := 1$.*

When it is clear that we are working within a Coxeter complex $\Sigma(W, S)$, we will simply denote a chamber $\{w\}$ as w. Each panel $w\langle s\rangle$ is the face of exactly two chambers, w and ws. In this case, we say that w and ws are s-adjacent. We have that W acts on Σ by left multiplication on the coset representatives, equivalent to left translation, so it is structure-preserving.

Notice that we have used the terminology of chamber complexes, without having even proved that Σ is a chamber complex, let alone a simplicial complex. We shall do that now.

Lemma 2.3 *Let (W, S) be a Coxeter system. The function $J \mapsto W_J$ is a poset isomorphism from the power set of S (ordered by inclusion) to the set of standard subgroups (ordered by inclusion). The inverse of this isomorphism is given by $W' \mapsto W' \cap S$, where W' is a standard subgroup.*

Proof. See [1], Proposition 2.13. ∎

Theorem 2.4 *A Coxeter complex $\Sigma(W, S)$ is a thin chamber complex with rank equal to $|S|$.*

Proof. We first show that Σ is a simplicial complex. We begin by showing any two elements $A, B \in \Sigma$ have a greatest lower bound. Let $A = w_1 W_J$ and $B = w_2 W_K$. By the structure-preserving action of W, this is equivalent to showing that $A' = w_1^{-1}w_1W_J = W_J$ and $B' = w_1^{-1}w_2W_K$ have a greatest lower bound. In other words, it is enough to consider W_J and wW_K for an arbitrary $w \in W$.

We will show that W_J and wW_K have a least upper bound in the set of standard cosets ordered by inclusion, which is equivalent to showing that they have a greatest lower bound when ordered by reverse inclusion. Any upper bound of both cosets will contain the identity element (inherited from W_J), so it will be a standard subgroup. To see this, if some standard coset $w'W_U$ contains the identity, this implies that w' is the inverse of some element in W_U. However, as W_U is already a subgroup, it contains w' as well then, hence w' acts as the identity on W_U. Furthermore, since an upper bound contains w, it also contains its inverse w^{-1}, hence the upper bound contains $W_K = w^{-1}wW_K$. Thus, an upper bound, which will be a standard subgroup, will contain the letters of K, J, and enough letters to represent w. By Proposition 1.5, there is a smallest subset $S(w)$ such that any reduced decomposition of w uses the letters in $S(w)$ and $W_{S(w)}$ is the smallest standard subgroup containing w. Thus, our least upper bound in the inclusion-ordered set of standard cosets is W_L where $L = J \cup K \cup S(w)$ (set-theoretic union), and hence, it is the greatest lower bound of W_S and wW_K in Σ.

Now we will prove that for any $A \in \Sigma$, the poset $\Sigma_{\leq A}$ of faces of A is isomorphic to the power set of $\{1, 2, \dots, r\}$ ordered by inclusion. If $A = wW_S$, then it is uniquely contained in the chamber $\{w\}$, that is, $A \in \Sigma_{\leq \{w\}}$. By the structure-preserving action of W, it is enough to consider the fundamental chamber $C = 1$. We have that $\Sigma_{\leq C}$ is the poset of standard subgroups, ordered by reverse inclusion, so by Lemma 2.3, $\Sigma_{\leq C} \cong (2^S)^{\mathrm{op}}$, where the latter is the power set of S ordered by reverse inclusion. We also have that $(2^S)^{\mathrm{op}} \cong 2^S$, where the latter is power set of S ordered by inclusion, by the map $J \mapsto S \setminus J$ for $J \subseteq S$. Furthermore, $2^S \cong 2^{\{1,2,\dots,|S|\}}$ by any map which sends elements of S uniquely to integer singeltons, so by a double application of isomorphism transitivity, $\Sigma_{\leq C} \cong 2^{\{1,2,\dots,|S|\}}$. Hence, for any $A \in \Sigma_{\leq C}$, we have that $\Sigma_{\leq A} \cong 2^{\{1,2,\dots,r\}}$, for some $r \leq |S|$, giving us the desired result. Thus, Σ is a simplicial complex.

Our above proof also demonstrates that all maximal simplices (chambers) of Σ have rank $|S|$. Furthermore, for any two chambers, we may build a gallery between them. To see this, we consider the general case of a gallery between C and $\{w\}$ where $w = s_1s_2 \dots s_d$. We have that

$$\Gamma : C, C_1, \dots, C_d = wC,$$

where $C_i = s_1s_2 \dots s_iC$ for $1 \leq i \leq d$, is a gallery between C and $\{w\}$ such that C_i is s_i-adjacent to C_{i-1}. By the action of W on Γ, we can see by this case that there is a gallery between any two chambers. Hence, Σ is a chamber complex. By construction, each panel is the face of exactly two chambers, so Σ is thin. ■

Proposition 2.5 *Σ is colorable with type function τ, such that for $wW_J \in \Sigma$, $\tau(wW_J) = S \setminus J$. Furthermore, the action of W is type-preserving.*

Proof. See [1], Theorem 3.5. ∎

The function τ is referred to as the *canonical type function* on Σ. Our "colors" associated with τ are the elements of S.

2.2 Foldings

In this section, we consider a special class of endomorphisms on Coxeter complexes known as *foldings*.

Proposition 2.6 *Let $\Sigma = \Sigma(W, S)$ be a Coxeter complex. For any two adjacent chambers C_1 and C_2, there is an endomorphism of Σ such that*

1. *ϕ is a retraction onto its image α.*
2. *Every chamber in α is the image of precisely one chamber in $\Sigma \setminus \alpha$.*
3. *$\phi(C_2) = C_1$.*

Proof. See [1], Proposition 3.38. ∎

We will return to such endormorphisms of Coxeter complexes. For now, we will deal with arbitrary thin chamber complexes.

Definition 2.7 *Let Σ be an arbitrary thin chamber complex. An endomorphism ϕ of Σ is* ***idempotent*** *if $\phi^2 = \phi$, that is, ϕ is a retraction onto its image. We call ϕ a* ***folding*** *if for every chamber $C \in \phi(\Sigma)$ there is precisely one chamber $C' \in \Sigma \setminus \phi(\Sigma)$ such that $\phi(C') = C$.*

Assume ϕ is a folding. Denoting $\alpha = \phi(\Sigma)$, we have that α is a simplicial complex and since ϕ take galleries to pregalleries, ϕ is a chamber complex. We already know that foldings exist for Coxeter complexes by Proposition 2.6.

Let α' be the chamber subcomplex of Σ consisting of chambers not in α and their faces. Respectively denoting the chambers of α and α' as $\mathcal{C}(\alpha)$ and $\mathcal{C}(\alpha')$, we have that $\phi|_{\mathcal{C}(\alpha')}$ is a bijection from $\mathcal{C}(\alpha')$ to $\mathcal{C}(\alpha)$. We will consider the function $\phi' : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)$, where $\mathcal{C}(\Sigma)$ consists of the chambers of Σ, where $\phi'|_{C(\alpha')} = \mathrm{id}$ (the identity map) and $\phi'|_{C(\alpha)}$ is the inverse of the bijection described above.

In an abstract geometric sense, α and α' partition Σ into equal halves, and ϕ "folds" $C(\alpha')$ and their faces onto $C(\alpha)$ and their faces. One may be inclined to interpret ϕ' as the "opposite folding" of ϕ, but this intuition is not entirely valid, as ϕ' has only been defined on chambers. If it so happens that ϕ' may be extended to a folding, then ϕ is referred to as a *reversible* folding, and the extension of ϕ', which for convenience we also denote as ϕ', is called the folding *opposite* to ϕ. In the setting of Coxeter complexes, every folding is reversible, but this is not the case for any thin chamber complex. Nevertheless, we will state a useful property of our originally defined ϕ' without assuming it has an extension.

Proposition 2.8 *Let Σ be a thin chamber complex and let ϕ be a folding of Σ, with α and α' as defined above. Then ϕ' takes adjacent chambers to chambers that are equal or adjacent. Furthermore, there exists a pair C, C' of adjacent chambers such that $C \in \alpha$ and $C \in \alpha'$, and for any such pair, $\phi(C') = C$ and $\phi'(C) = C'$.*

Proof. See [1], Lemma 3.42 and Lemma 3.43. ∎

What this tells us is that if we were to take a minimal gallery Γ from a chamber in α to a chamber in α', we would have to cross a "wall" separating two adjacent chambers. Then $\phi(\Gamma)$, which would fold Γ entirely onto α, would be a pregallery with a consecutive repetition at the points of the sequence corresponding to the adjacent chambers.

For the following resullts, we assume Σ is a thin chamber complex with folding ϕ, and associated chamber partitions α and α'.

Lemma 2.9 *If Γ is a minimal gallery with extremeties in α (or α'), then Γ lies entirely in α.*

Proof. Let $\Gamma : C_0, C_1, \dots, C_d$ be a minimal gallery with $C_0, C_d \in \alpha$. If Γ does not lie entirely in α, then it must pass from α to α' at some point in the sequence. That is, there is a pair of adjacent chambers $C_i \in \alpha$, and $C_{i+1} \in \alpha'$. We have then by Propositon 2.8, that $\phi(\Gamma)$ is a pregallery with the same extremities and length of Γ, but containing a consecutive repetition $\phi(C_i) = \phi(C_{i+1})$, so we may shorten $\phi(\Gamma)$, contradicting the minimality of Γ. Thus, Γ lies entirely in α. By a very similar proof, we may show the same result for extremities lying in α'. ∎

Lemma 2.10 *Suppose C and C' are adjacent chambers such that that $\phi(C') = C$. Then ϕ is the unique folding taking C' to C.*

Proof. See [1], Lemma 3.46. ∎

We now arrive at the main result of reversible foldings,

Theorem 2.11 *Let Σ be a thin chamber complex with folding ϕ, and associated chamber partitions α and α'. If $\phi(C') = C$ for some $C \in \alpha$ and $C' \in \alpha'$, then ϕ is reversible if and only if there exists a folding which takes C to C'. In particular, if ϕ is reversible with opposite folding ϕ', then there is an automorphism s of Σ such that $s|_\alpha = \phi'$ and $s|_{\alpha'} = \phi$.*

Proof. See [1], Lemma 3.49. ∎

We also have that s is order 2 and is completely characterizable as the non-trivial automorphism which fixes the vertices of the panel $C \cap C'$. Furthemore, the set of of simplices of Σ fixed by s is the subcomplex $\alpha \cap \alpha'$, which should evoke the image of a "wall" we discussed previously.

Definition 2.12 *Let Σ be a thin chamber complex. A* ***root*** *is a subcomplex α which is the image of a reversible folding ϕ. The subcomplex α' generated by chambers not in α, which is the image of the opposite folding ϕ', is also called a root, and is often denoted as $-\alpha$. We call the intersection $H = \alpha \cap -\alpha$, the* ***wall*** *bounding $\pm\alpha$.*

A wall H is associated with an automorphism $s := s_H$ by theorem 2.11, which interchanges the roots α and $-\alpha$ on either side of H, and fixes every simplex of H. As such, we call s_H the reflection of Σ with respect to H.

In the context of Coxeter complexes, Proposition 2.6 abstractly states that any two adjacent chambers C_1, C_2 are separated by a wall, and ϕ is the unique folding taking C_2 to C_1. As we stated before, every folding of a Coxeter complex $\Sigma(W, S)$ is reversible, so the opposite folding ϕ' is the unique folding taking C_1 to C_2. We have then that C_1 and C_2 determine a wall, and a reflection $t \in W$ which acts by left multiplication.

Lemma 2.13 *Let $\Sigma(W, S)$ be a Coxeter complex. The set of reflections T is characterized by the left action of elements of the form*

$$T = \{t = usu^{-1} \mid u \in W, s \in S\}.$$

Proof. By Proposition 2.6, any two adjacent chambers in a Coxeter complex will be separated by a wall. If $C_1 = uC$ and $C_2 = usC$, for $C = 1$ the fundamental chamber, $u \in W$, and $s \in S$, then a reflection associated to the wall between C_1 and C_2 is the left action of $t = usu^{-1}$. By Lemma 2.10, such reflection determines the unique folding and opposite folding which will take C_1 and C_2 to one another, so in fact, $T = \{t = usu^{-1} \mid u \in W, s \in S\}$ forms the complete set of reflections described in Theorem 2.11. ∎

2.3 The Bruhat Order

In this last section, we will use the geometry of Coxeter complexes to prove the *strong exchange condition*, which is otherwise difficult to prove using only algebraic arguments. We will then introduce a partial ordering on a Coxeter system known as the *Bruhat ordering*. We first prove the following,

Lemma 2.14 *Let Σ be a thin chamber complex. Let α be a root with s the associated reflection. If C and C' are chambers in α, then $d(C, sC') > d(C, C')$, where $d(-,-)$ is the gallery distance defined in §1.3.*

Proof. Let ϕ be the associated folding of $-\alpha$ onto α and H the wall determined by $\pm\alpha$. We have then that C and sC' are separated by H. Now let $\Gamma : C_0 = C, C_1, \ldots, C_d = sC'$ be a minimal gallery from C to sC'. The gallery must cross H at some point, that is, there is a pair of adjacent chambers C_i, C_{i+1} separated by H such that $C_i = \phi(C_{i+1})$ by Proposition 2.8. We have that $\phi(\Gamma)$ is a gallery from C to C', with the same length as Γ, so $d(C, sC') \geq d(C, C')$. However, since there is a repetition in $\phi(\Gamma)$, we may obtain a shorter gallery from C to C', hence $d(C, sC') > d(C, C')$. ■

Let (W, S) be a Coxeter system. The *exchange condition* states that given $w \in W$, $s \in S$, and a reduced decomposition $w = s_1 \ldots s_d$ of w, we have that $l(sw) = d + 1$ or else there is an index i such that $w = ss_1 \ldots \hat{s}_i \ldots s_d$.

Proposition 2.15 *Let (W, S) be a Coxeter system. Given $w \in W$, suppose t is a reflection such that $l(tw) < l(w)$. Then for any S-decomposition $w = s_1 \ldots s_d$, there is an index i such that $tw = s_1 \ldots \hat{s}_i \ldots s_d$ for some index $1 \leq i \leq d$. This is referred to as the strong exchange condition.*

Proof. Set $C = 1$ as the fundamental chamber of $\Sigma = \Sigma(W, S)$. Let H be the wall associated to t. We first show that C and wC are separated by H. Assuming otherwise, we have that $d(C, twC) > d(C, wC)$ by Lemma 2.14. However, the length of a minimal gallery from $C = 1$ to $w'C$ for any $w' \in W$ is equal to $l(w')$ by an earlier remark. Thus, if C and wC were on the same side of H, this would imply $l(tw) > l(w)$ contrary to our assumption, so C and wC are separated by H. Consider the gallery

$$\Gamma : C_0 = C, C_1, \ldots, C_d = wC$$

where $C_i = s_1 \ldots s_i C$. By our previous argument, Γ crosses H, that is, there is a pair of adjacent chambers C_{i-1}, C_i such that $C_{i-1} = tC_i$. This implies that $s_1 \ldots s_{i-1} = ts_1 \ldots s_i$, so we have an explicit representation

$$t = (s_1 \ldots s_{i-1})s_i(s_{i-1} \ldots s_1).$$

So,

$$tw = (s_1 \ldots s_{i-1})s_i(s_{i-1} \ldots s_1)(s_1 \ldots s_d) = s_1 \ldots \hat{s}_i \ldots s_d$$

■

Proposition 2.16 *Let (W, S) be a Coxeter system. Given $w, w' \in W$, the following conditions are equivalent*

1. *For every decomposition $w = s_1 \ldots s_d$, there is an index i such that $w' = s_1 \ldots \hat{s}_i \ldots s_d$.*
2. *For some reduced decomposition $w = s_1 \ldots s_d$, there is an index i such that $w' = s_1 \ldots \hat{s}_i \ldots s_d$.*
3. *$l(w') < l(w)$, and there is a reflection t such that $w' = tw$.*

Proof. $1 \implies 2$ is trivial. For $2 \implies 3$, assume for some reduced decomposition $w = s_1 \dots s_d$, there is an index i such that $w' = s_1 \dots \hat{s}_i \dots s_d$. Clearly then $l(w') < l(w)$. Now let $t = ms_im^{-1}$ where $m = s_1 \dots s_{i-1}$. We have then that $w' = tw$. Lastly, for $2 \implies 3$, assume $l(w') < l(w)$ and there is a reflection t such $w' = tw$. This implies that $l(tw) < l(w)$, so by the strong exchange condition there is an index i such that if $w = s_1 \dots s_d$ then $tw = s_1 \dots \hat{s}_i \dots s_d$. ■

The *Bruhat graph* of a Coxeter system (W, S) is the directed graph with vertex set W and a directed edge $w' \to w$, for $w, w' \in W$, whenever w' and w satisfy the equivalent conditions in Proposition 2.16. This defines a partial order on W, called the *Bruhat order*, with $w' \leq w$ if and only if there is a directed path

$$w' = w_0 \to w_1 \to \cdots \to w_k = w$$

from w' to w ($k \geq 0$). A directed cycle is a directed path such that the initial and terminal vertices of the path are equal.

Lemma 2.17 *The Bruhat graph is acyclic, meaning there are no directed cycles.*

Proof. We first note that the Bruhat order is transitive. If $w \leq w'$ and $w' \leq w''$, then we can extend the directed path from w to w' to include the edges from w' to w'' by invoking Proposition 2.16, hence $w \leq w''$. If $w_1, w_2, \dots, w_k$ form a direct cycle, then

$$w_1 \leq w_3 \leq \cdots \leq w_k \leq w_1$$

So we have that $w_1 \leq w_k$ and $w_k \leq w_1$. If $w_1 \neq w_k$, then we have that $(w_1 < w_k)$, which by condition 2 in Proposition 2.16 implies that for some reduced decomposition of w_k, we may delete one letter from it to produce w_1. In the other direction $(w_k < w_1)$, by condition 1 of Proposition 2.16, for the decomposition of w_1 produced by the first process, w_k may be produced by deleting a letter from w_1. However, this is contradictory, since we assumed the original decomposition of w_k was reduced. Thus, there are no cycles. Furthermore, this demonstrates that the Bruhat order is antisymmetric. ■

Let (W, S) be a Coxeter system. Given $w \in W$ and $s \in S$, we have that sw and w are always comparable in the Bruhat order, since $w < sw$ if $l(sw) > l(w)$ and $sw < w$ otherwise. This gives us the meaningful expression $\max\{sw, w\}$.

Proposition 2.18 *For a Coxeter system (W, S) and $w, w' \in W$, if $w' < w$ then $sw' \leq max\{sw, w\}$ for any $s \in S$.*

Proof. We begin by assuming there is a directed edge $w' \to w$. Let $w = s_1 \dots s_d$ be a reduced decomposition. So by condition 2 of Proposition 2.16, $w' = s_1 \dots \hat{s}_i \dots s_d$ for some index i. If $l(sw) > l(w)$, this implies that sw is reduced. We have that $sw' = ss_1 \dots \hat{s}_i \dots s_d$, so $sw' < sw = max\{w, sw\}$.

In the case that $l(sw) < l(w)$, we can assume without loss of generality that we are working with decomposition of w such that $s = s_1$. We will now consider the index i such that s_i is deleted from w to produce w'. If $i = 1$, then $sw' = s_1 w' = s_1 \hat{s_1} \dots s_d = w$, so $sw' = w = max\{sw, w\}$. If $i > 1$, then

$$sw' = s_1 s_1 \dots \hat{s_i} \dots s_d = s_2 \dots \hat{s_i} \dots s_d$$

By setting $w'' = s_2 s_3 \dots s_d$, it is clear that w'' is reduced, so we have a directed edge $sw' \to w''$. Since w'' is obtainable by deleting s_1 from the reduced decomposition of w, we have $w'' \to w$. Thus, we have

$$sw' \to w'' \to w.$$

So by transitivity, $sw' < w = \max\{w, sw\}$.

Now assume that w' and w are path-connected, but not edge-connected. That is,

$$w' = w_1 \to w_2 \to \cdots \to w_k = w.$$

By the edge-connected case,

$$sw' = sw_1 \leq \max\{sw_1, w_1\} \leq \cdots \leq \max\{sw_k, w_k\} = \max\{sw, w\}.$$

Thus, by transitivity, $sw \leq \max\{sw, w\}$. ■

Proposition 2.19 *Let (W, S) be a Coxeter system. Given $w, w' \in W$, the following conditions are equivalent:*

1. *$w' < w$ in the Bruhat order.*
2. *For every decomposition of w as a product of elements of S, there is a decomposition of w' obtained by deleting one or more letters.*
3. *For some reduced decomposition of w, there is a decomposition of w' obtained by deleting one or more letters.*

Proof. To show $1 \implies 2$, first let $w = s_1 \dots s_d$ be an arbitrary decomposition. Since $w' < w$, there is a path

$$w' = w_1 \to w_2 \to \cdots \to w_k = w$$

Moving from right to left down the path, we begin with the chosen decomposition of $w = w_k$, and repeatedly apply condition 1 of Proposition 2.16, to arrive at a decomposition of w' which is the result of deleting one or more letters from our original decomposition of w.

$2 \implies 3$ is trivial, so we are left with proving $3 \implies 1$. Let $w = s_1 \dots s_d$ be a reduced decomposition and assume w' is obtained by deleting one or more letters. We will use strong induction on $l(w) = d$ to show that $w' < w$. Let $g = s_2 \dots s_d$, so clearly $g < w$. In the base case, $l(w) = 1$, there is only one possible letter to delete and clearly $w' = 1 < w$. So assume for $l(w) < d$ our desired conclusion holds, and we will prove our conclusion also holds for $l(w) = d$. Since g is reduced and $g < w$, if we assume that w' is produced by deleting s_1 and 0 or more other letters from w, then w' is produced by deleting 0 or more letters from g. Hence by induction we have that $w' \leq g$, thus by transitivity $w' < w$. Assume w' is produced by deleting letters other than s_1 from w. That is, $w' = s_1 g'$, where g' is the result of deleting letters from g. We have by induction that $g' < g$, so by Proposition 2.18, $w' \leq \max\{s_1 g, g\} = s_1 g = w$. We have that $l(w') < l(w)$, thus $w' < w$, which completes our proof by induction.

■

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